Least-squares methods for the velocity-pressure-stress formulation of the Stokes equations[1]

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Abstract

We formulate and study finite element methods for the solution of the incompressible Stokes equations based on the application of least-squares minimization principle to an equivalent first order velocity-pressure-stress system. Our least-squares functional involves the L^2-norms of the residuals of each equation multiplied by a mesh dependent weight. Each weight is determined according to the Agmon-Douglis-Nirenberg index of the corresponding equation. As a result, the approximating spaces are not subject to the LBB condition and conforming discretizations are possible with merely continuous finite element spaces. Moreover, the resulting discrete problems involve only symmetric, positive definite systems of linear equations, *i.e.*, assembly of the discretization matrix is not required even at the element level. We prove that the least-squares approximations converge to the solutions of the Stokes problem at the best possible rate and then present some numerical examples illustrating our theoretical results. Among other things, these numerical examples indicate that the method is not optimal without the weights in the least-squares functional .

1 Introduction

Recently there has been an increased interest in the application of least-squares ideas for the approximate solution of flow problems, see *e.g.* [4], [5], [6], [7], [9], [10], [12], [13], [19], [20], [21], [22], [24]. In contrast with the mixed methods, the variational problem in the least-squares approach is derived as the necessary condition (Euler-Lagrange equation) for the minimizers of a suitably defined quadratic functional involving residuals of the differential equations. Thus, a least-squares finite element method is based on discretization of a minimization problem rather than a saddle point optimization problem. As a result, stability of the least-squares finite element method can be guaranteed by the simple inclusion of the discretization spaces into the respective continuous

[1]This work was supported by the Air Force Office of Scientific Research under grants number AFOSR-93-1-0061 and number AFOSR-93-1-0280

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spaces, *i.e.*, inf-sup or LBB type of conditions (see [16] or [17]), are not required. In particular, one can use the same order of interpolation for all unknowns. Moreover, the application of the least-squares minimization principle results in discrete problems with symmetric and positive definite matrices. Solution of these problems can be accomplished by robust and efficient iterative methods (like conjugate gradient methods). As a result, a method can be devised that is assembly free even at the element level.

Among the first least-squares methods proposed for the Stokes equations is the method of Aziz et. al. [2]. In this method the least-squares functional is formulated for the primitive variable form of the Stokes equations. However, from the practical point of view it is more advantageous first to transform the Stokes problem into an equivalent first order system and then to apply the least-squares minimization principle to this system. Indeed, if the least-squares functional contains only L^2-norms of the residuals of the equations and if the highest order of differentiation in each equation does not exceed one then it is possible to obtain conforming discretizations of the variational problem (the Euler-Lagrange equation) using simple continuous finite element spaces. So far this approach has been applied succesfully to the Stokes equations cast into first order systems involving the velocity, vorticity and the pressure, [5], [7], [10], [13] and the accleration and the pressure [9] as the dependent variables.

Here we shall be concerned chiefly with the formulation and analyses of least-squares methods for the incompressible Stokes equations based on an equivalent formulation involving the velocity, pressure and the extra stress tensor as the dependent variables. Mixed methods for the Navier-Stokes equations based on velocity-pressure-stress formulation have been considered in [23]. Formulations involving the stress as a primary variable have also been used in the context of the stabilized Galerkin methods for the Stokes and the linearized Navier-Stokes equations in [3] and [15]. In contrast here we formulate a *bona fide* least-squares method. In the course of the formulation and the analyses we shall compare the resulting method with a similar one based on the velocity-vorticity-pressure form of the Stokes problem [4], [5]. Both the velocity-pressure-stress and the velocity-vorticity-pressure equations are equivalent with the Stokes problem in primitive variables, however, from the point of view of the elliptic regularity theory of [1] there are certain distinctions between the two forms of the Stokes problem.

For other applications of the least-squares ideas for the approximate solution of elliptic partial differential equations we refer the interested reader to [2], [8], [27] and [28]. For standard Galerkin methods based on the velocity-vorticity formulation one may also consult [18].

The paper is organized as follows. In the next section we introduce the velocity-pressure-stress Stokes system. Then in Section 3 we state briefly some results from the Agmon, Douglis and Nirenberg (ADN) [1] elliptic regularity theory which will be needed for the error estimates. In that section we also extend the ADN apriori estimate for the velocity-pressure-stress equations to negative regularity indices. The mesh dependent (or *weighted*) least-squares functional, the corresponding variational problem and the error estimates of the respective weighted least-squares approximations are the subject of Section 4. In this section we also derive upper bounds for the condition numbers of the discretization matrices. Finaly, in Section 5 we present results of some numerical experiments with the least-squares finite element method defined in Section 4. Our experiments demonstrate that all unknowns, including the stress tensor components, are approximated optimally by the weighted method and that the convergence rates are reduced by approximately one order of accuracy if the weights are removed from the least-squares functional. In addition, our experiments suggest that the condition number estimates of Section 4 might be overly pessimistic.

2 Velocity-Pressure-Stress Equations

We consider the incompressible Stokes equations

$$
\begin{aligned}
-\nu \triangle \mathbf{u} + \nabla p &= \mathbf{f} \ \text{ in } \Omega \\
\nabla \cdot \mathbf{u} &= 0 \ \text{ in } \Omega\,,
\end{aligned}
\tag{1}
$$

where $\Omega \in \mathbf{R}^n, n = 2, 3$, is open and bounded set with a smooth boundary Γ, $\mathbf{u}$ denotes the velocity, p the pressure, and $\mathbf{f}$ is the body force. The system (1) is uniformly elliptic and its total order is four or six in two or three space dimensions respectively (see Section 3 for a precise definition of uniform ellipticity and total order). Equations (1) must be supplemented with boundary conditions and here we shall consider the velocity boundary condition

$$
\mathbf{u} = \mathbf{u}_0 \ \text{ on } \Gamma\,. \tag{2}
$$

Let $\epsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ denote the symmetric part of the velocity gradient, *i.e.*, the deformation tensor. The extra stress tensor T is defined as

$$
T = 2\nu\, \epsilon(\mathbf{u})\,. \tag{3}
$$

In view of the incompressibility constraint and the vector identity

$$
\nabla \cdot T = \nu(\triangle \mathbf{u} + \nabla(\nabla \cdot \mathbf{u}))
$$

we can replace the first equation in (1) by $-\nabla \cdot T + \nabla p = \mathbf{f}$. Thus we have the following first order velocity-pressure-stress formulation of the Stokes problem:

$$
\begin{aligned}
T - 2\nu\, \epsilon(\mathbf{u}) &= 0 \ \text{ in } \Omega \\
-\nabla \cdot T + \nabla p &= \mathbf{f} \ \text{ in } \Omega \\
\nabla \cdot \mathbf{u} &= 0 \ \text{ in } \Omega \\
\mathbf{u} &= \mathbf{u}_0 \ \text{ on } \Gamma\,.
\end{aligned}
\tag{4}
$$

In two dimensions (4) is a system of six unknowns and six equations and in three dimensions the number of the unknowns and the equations increases to ten.

Our analysis of the least-squares methods for the velocity-pressure-stress equations (4) will be greatly facilitated if (4) is a self-adjoint boundary value problem. However it is easy to see that (4) is not self-adjoint in the physical variables defined above. To deliver a velocity-pressure-stress formulation with the desired property we first redefine the stress tensor (3) as

$$
T = \sqrt{2\nu}\, \epsilon(\mathbf{u}). \tag{5}
$$

Then we multiply the momentum equation in (4) by -1 and permute the continuity equation and the momentum equation which yields the following generalized velocity-pressure-stress system

$$\begin{aligned} T - \sqrt{2\nu}\,\epsilon(\mathbf{u}) &= F_1 \text{ in } \Omega \\ \nabla\cdot\mathbf{u} &= f_2 \text{ in } \Omega \\ \sqrt{2\nu}\,\nabla\cdot T - \nabla p &= \mathbf{f}_3 \text{ in } \Omega \\ \mathbf{u} &= \mathbf{u}_0 \text{ on } \Gamma\,. \end{aligned} \tag{6}$$

Using the identities

$$\int_\Omega (\nabla\mathbf{u} + \nabla\mathbf{u}^T) : D\,dx = -2\int_\Omega (\nabla\cdot D)\cdot\mathbf{u}\,dx$$

$$\int_\Omega (\nabla\cdot T)\cdot\mathbf{v}\,dx = -\frac{1}{2}\int_\Omega (\nabla\mathbf{v} + \nabla\mathbf{v}^T) : T\,dx$$

which hold for any symmetric tensors D and T and vector functions $\mathbf{u}, \mathbf{v}$ vanishing on the boundary, it is easy to show that the problem (6) is self-adjoint. Evidently, if the tensor F_1 and the function f_2 are identicaly zero the Stokes problem (1) is equivalent to the generalized system (6). If the generalized equations are considered on their own, then the function f_2 must be subject to the following solvability constraint:

$$\int_\Omega f_2(x)\,dx = \int_\Gamma \mathbf{u}_0\cdot\mathbf{n}\,dx\,. \tag{7}$$

For simplicity we shall formulate our least-squares methods under the assumption that the boundary condition can be satisfied exactly. Then without loss of generality we may assume homogeneous boundary condition, *i.e.*, $\mathbf{u}_0 = 0$ in which case (7) means that f_2 must have zero mean. This assumption turns out to be quite convenient for the purposes of the error analysis of the least-squares methods.

To guarantee the uniqueness of the solutions of (6) one has to impose an additional constraint on the pressure. The customary choice is to require that the pressure have zero mean over Ω, *i.e.*,

$$\int_\Omega p\,dx = 0\,. \tag{8}$$

Under this assumption one can show the following.

Proposition 1 *The problem* (6)-(8) *has unique solution for all smooth data* $F_1, f_2, \mathbf{f}_3$ *and* $\mathbf{u}_0$

Proof. Assume that T, p and $\mathbf{u}$ are smooth functions that satisfy the system (6) and conditions (7), (8) for $F_1 = 0$, $f_2 = 0$, $\mathbf{f}_3 = 0$ and $\mathbf{u}_0 = 0$. From the first equation $T = \sqrt{2\nu}\epsilon(\mathbf{u})$. Then we replace T in the next to last equation in (6) to find that the pair $\mathbf{u}$, p satisfies the homogeneous Stokes problem

$$\begin{aligned} -\nu\triangle\mathbf{u} + \nabla p &= \mathbf{0} \text{ in } \Omega \\ \nabla\cdot\mathbf{u} &= 0 \text{ in } \Omega \\ \mathbf{u} &= \mathbf{0} \text{ on } \Gamma\,. \end{aligned}$$

From the well-known uniqueness result for the Stokes problem (see [16]) we can conclude that $p = 0$, $\mathbf{u} = \mathbf{0}$ and $T = 0$. $\Box$

In order to give proper formulation of our least-squares methods we need to introduce the necessary function spaces. We use $\mathcal{D}(\Omega)$ to denote the space of smooth functions with compact support in Ω and $\mathcal{D}(\bar{\Omega})$ to denote the restrictions of the functions in $\mathcal{D}(\mathbf{R}^n)$ on $\bar{\Omega}$. For $s \geq 0$ we use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma)$ with inner products and norms denoted by $(\cdot,\cdot)_{s,\Omega}$ and $(\cdot,\cdot)_{s,\Gamma}$ and $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\Gamma}$, respectively. Often, when there is no chance for confusion, we will omit the measure Ω from the inner product and norm designation.

As usual, $H_0^s(\Omega)$ will denote the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$ and $L_0^2(\Omega)$ will denote the subspace of square integrable functions with zero mean. We set $\tilde{\mathcal{D}}(\Omega) = \mathcal{D}(\Omega) \cap L_0^2(\Omega)$, $\tilde{\mathcal{D}}(\bar{\Omega}) = \mathcal{D}(\bar{\Omega}) \cap L_0^2(\Omega)$ and $\tilde{H}^s(\Omega) = H^s(\Omega) \cap L_0^2(\Omega)$. For negative values of s the spaces $H^s(\Omega)$, $H_0^s(\Omega)$ and $\tilde{H}^s(\Omega)$ are defined as the closures of $\mathcal{D}(\bar{\Omega})$, $\mathcal{D}(\Omega)$ and $\tilde{\mathcal{D}}(\bar{\Omega})$ with respect to the norm

$$\|\phi\|_s = \sup_{q \in D(\Omega)} \frac{\int_\Omega \phi q \, dx}{\|q\|_{-s}}; \tag{9}$$

where $D(\Omega) = \mathcal{D}(\bar{\Omega}), \mathcal{D}(\Omega)$ and $\tilde{\mathcal{D}}(\bar{\Omega})$ respectively. We identify $H^s(\Omega)$, $H_0^s(\Omega)$ and $\tilde{H}^s(\Omega)$ with the duals of $H^{-s}(\Omega)$, $H_0^{-s}(\Omega)$ and $\tilde{H}^{-s}(\Omega)$ respectively; for $s \in \mathbf{R}$ these spaces form interpolating families. By $(\cdot,\cdot)_{(s_1,\dots,s_n)}$ and $\|\cdot\|_{(s_1,\dots,s_n)}$ we denote inner products and norms, respectively, on the product spaces $H^{s_1}(\Omega) \times \cdots \times H^{s_n}(\Omega)$; when all s_i are equal we shall simply write $(\cdot,\cdot)_{s,\Omega}$ and $\|\cdot\|_{s,\Omega}$. Finally, we use C to denote a generic constant.

3 The Agmon-Douglis-Nirenberg estimates

In this section we derive the a priori estimates for the problem (6) that are relevant for the analysis of the least-squares methods. First, we discuss the Agmon-Douglis-Nirenberg (ADN) theory for elliptic boundary value problems. We define ellipticity and the uniform ellipticity of systems of PDEs following ADN [1] and introduce the various conditions that are necessary for the a priori estimates of [1] to hold. Then, we verify the conditions of [1] for the velocity-pressure-stress system (6) and state the a priori estimates. Finally, these estimates are extended over negative Sobolev spaces.

3.1 The complementing condition

Let $\mathcal{L} = \{\mathcal{L}_{ij}\}$, $i,j = 1,\dots,N$, denote a differential operator and let $\mathcal{R} = \{\mathcal{R}_{lj}\}$, $l = 1,\dots,m$, $j = 1,\dots,N$, denote a boundary operator. Consider a general boundary value problem of the form

$$\mathcal{L}U = F \quad \text{in } \Omega \tag{10}$$

$$\mathcal{R}U = G \quad \text{on } \Gamma\,. \tag{11}$$

We assign a system of integer indices $\{s_i\}$, $s_i \leq 0$, for the equations and $\{t_j\}$, $t_j \geq 0$, for the unknown functions, such that the order of $\mathcal{L}_{ij}$ is bounded by $s_i + t_j$. The principal part $\mathcal{L}^p$ of $\mathcal{L}$ is defined as all those terms $\mathcal{L}_{ij}$ with orders exactly equal to $s_i + t_j$. The principal part $\mathcal{R}^p$ is defined in a similar way by assigning nonpositive weights r_l to each row in $\mathcal{R}$ such that the order of $\mathcal{R}$ is bounded by $r_l + t_j$. We shall say that $\mathcal{L}$ is *elliptic* of total order $2m$ if there exists a set of indices t_j and s_i and a positive integer m such that

$$\det \mathcal{L}^p(\boldsymbol{\xi}) \neq 0 \quad \text{for all real } \boldsymbol{\xi} \neq 0$$

and $\deg(\det \mathcal{L}^p(\boldsymbol{\xi})) = 2m$. We shall say that $\mathcal{L}$ is *uniformly elliptic* if in addition there exists a constant C_e, such that

$$C_e^{-1}|\boldsymbol{\xi}|^{2m} \leq |\det \mathcal{L}^p(\boldsymbol{\xi})| \leq C_e|\boldsymbol{\xi}|^{2m} . \tag{12}$$

The Agmon-Douglis-Nirenberg theory permits nonuniqueness of the principal part, *i.e.*, it is possible to have more than one set of indices such that the above inequalities are satisfied, see [5].

Let us now discuss conditions on the operators $\mathcal{L}$ and $\mathcal{R}$ that will guarantee the well-posedness of the boundary value problem (10), (11). We assume that $\mathcal{L}$ is elliptic of total order $2m$. The first condition is to require that the number of rows in $\mathcal{R}$ equals m. For example, the Stokes operator in (1) is of total order four in two-dimensions and of total order six in three-dimensions. Therefore a boundary operator for (1) should prescribe two and three conditions on the boundary Γ in two and three-dimensions, respectively. Note that the velocity boundary operator (2) satisfies this condition. Second, we require that the following condition is satisfied.

> **Supplementary Condition on $\mathcal{L}$.** *First,* $\det \mathcal{L}^p(\boldsymbol{\xi})$ *is of even degree* $2m$ *(with respect to* $\boldsymbol{\xi}$*). Also, for every pair of linearly independent real vectors* $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, *the polynomial* $\det \mathcal{L}^p(\boldsymbol{\xi} + \tau\boldsymbol{\xi}')$ *in the complex variable* τ *has exactly* m *roots with positive imaginary part.*

For any elliptic system in three or more dimensions, the supplementary condition is satisfied, *i.e.*, the characteristic equation $\det \mathcal{L}^p(\boldsymbol{\xi} + \tau\boldsymbol{\xi}') = 0$ always has exactly m roots with positive imaginary parts. In two-dimensions, this condition must be verified for any given $\mathcal{L}^p$.

The last, third condition, is the celebrated complementing condition. It is a local algebraic condition on the principal parts $\mathcal{L}^p$ and $\mathcal{R}^p$ of the differential and boundary operators which guarantees the compatibility of a particular set of boundary conditions with the given system of differential equations. This condition is necessary and sufficient for coercivity estimates to be valid; see [1]. Before introducing the complementing condition, some notation must be established. Let $\tau_k^+(\boldsymbol{\xi})$ denote the m roots of $\det \mathcal{L}^p(\boldsymbol{\xi} + \tau\boldsymbol{\xi}')$ having positive imaginary part. Let

$$M^+(\boldsymbol{\xi}, \tau) = \prod_{k=1}^{m} \left(\tau - \tau_k^+(\boldsymbol{\xi})\right)$$

and let $\mathcal{L}'$ denote the adjoint matrix to $\mathcal{L}^p$. Then, we have the following definition [1].

> **Complementing condition.** *For any point* $P \in \Gamma$ *let* $\mathbf{n}$ *denote the unit outward normal vector to the boundary* Γ *at the point* P. *Then, for any real, non-zero vector* $\boldsymbol{\xi}$ *tangent to* Γ *at* P, *regard* $M^+(\boldsymbol{\xi}, \tau)$ *and the elements of the matrix*
>
> $$\sum_{j=1}^{N} \mathcal{R}_{lj}^p(\boldsymbol{\xi} + \tau\mathbf{n})\mathcal{L}'_{jk}(\boldsymbol{\xi} + \tau\mathbf{n})$$

as polynomials in τ. The operators $\mathcal{L}$ and $\mathcal{R}$ satisfy the complementing condition if the rows of the latter matrix are linearly independent modulo $M^{+}(\boldsymbol{\xi},\tau)$, i.e.,

$$\sum_{l=1}^{m} C_l \sum_{j=1}^{N} \mathcal{R}^p_{lj}\mathcal{L}'_{jk} \equiv 0 \pmod{M^+} \tag{13}$$

if and only if the constants C_l all vanish.

In [1], the following result is proved.

Theorem 1 *Assume that the system $\mathcal{L}U = F$ is uniformly elliptic (and in 2D satisfies the Supplementary Condition) and assume that the boundary operator $\mathcal{R}U$ satisfies the Complementing Condition. Furthermore, assume that for some $q \geq 0$, $U \in \prod_{j=1}^{N} H^{q+t_j}(\Omega)$, $F \in \prod_{i=1}^{N} H^{q-s_i}(\Omega)$, and $G \in \prod_{l=1}^{m} H^{q-r_l-1/2}(\Gamma)$. Then, there exists a constant $C > 0$ such that*

$$\sum_{j=1}^{N} \|u_j\|_{q+t_j,\Omega} \leq C \left(\sum_{i=1}^{N} \|F_i\|_{q-s_i,\Omega} + \sum_{l=1}^{m} \|G_l\|_{q-r_l-1/2,\Gamma} + \sum_{j=1}^{N} \|u_j\|_{0,\Omega} \right) . \tag{14}$$

Moreover, if the problem $\mathcal{L}U = F$, $\mathcal{R}U = G$ has a unique solution in the indicated spaces, then the L^2-norm on the right-hand side of (14) *can be omitted.*

From Theorem 1 it is clear that the indices t_j and s_i define the spaces for the unknowns and the data, respectively, in which the boundary value problem (10), (11) is well-posed. If all t_j can be chosen equal we shall say that the a priori estimate (14) holds under the assumption of an equal order of differentiability. Otherwise, we shall say that (14) holds under the assumption of a different order of differentiability.

3.2 The a priori estimates for the velocity-pressure-stress equations

For the sake of brevity we shall limit our discussion to the case of two-dimensions. In three-dimensions derivation of the a priori estimates is almost identical but the technical details involved in the verification of the various conditions and in particular, in the verification of the complementing condition, are formidable. In two-dimensions we assume that the unknowns are ordered as:

$$U = (T_1, T_2, T_3, p, u_1, u_2)\,,$$

where $T_1 = T_{11}$, $T_2 = T_{12}$ and $T_3 = T_{22}$ and that the six differential equations in (6) are ordered as

$$\mathcal{L}U = \begin{pmatrix} T_1 - \sqrt{2\nu}\frac{\partial u_1}{\partial x} \\ 2T_2 - \sqrt{2\nu}(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}) \\ T_3 - \sqrt{2\nu}\frac{\partial u_2}{\partial y} \\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \\ \sqrt{2\nu}(\frac{\partial T_1}{\partial x} + \frac{\partial T_2}{\partial y}) - \frac{\partial p}{\partial x} \\ \sqrt{2\nu}(\frac{\partial T_2}{\partial x} + \frac{\partial T_3}{\partial y}) - \frac{\partial p}{\partial y} \end{pmatrix} . \tag{15}$$

According to these ordering agreements we choose the following indices

$$t_1 = t_2 = t_3 = t_4 = 1, t_5 = t_6 = 2$$

$$s_1 = s_2 = s_3 = s_4 = -1, s_5 = s_6 = 0$$

for the unknowns and the differential equations, respectively. For this choice of indices we have that

$$\mathcal{L}^P = \mathcal{L}\,,$$

where $\mathcal{L}$ is defined in (15) and that

$$\det \mathcal{L}^p(\boldsymbol{\xi}) = \det \mathcal{L}(\boldsymbol{\xi}) = -\nu(\xi_1^2 + \xi_2^2)^2 = -\nu|\boldsymbol{\xi}|^4\,.$$

As a result, the uniform ellipticity condition

$$C_e^{-1}|\boldsymbol{\xi}|^{2m} \le |\det \mathcal{L}^p(\boldsymbol{\xi})| \le C_e|\boldsymbol{\xi}|^{2m}$$

holds with $m = 2$ and $C_e = \nu$. In other words, the velocity-pressure-stress system in two-dimensions is uniformly elliptic of total order four and one must specify two conditions on the boundary Γ. This total order is the same as for the Stokes problem (1) in the primitive variables and therefore, one can use the same boundary operator (2). The boundary operator (2) does not involve differentiation and the choice of $t_5 = t_6 = 2$ implies that one has to take $r_1 = r_2 = -2$. Finally, it is also easy to see that $\mathcal{L}^p$ satisfies the supplementary condition.

Note that the choice of t_js above implies different orders of differentiability for the pressure and the stress components and the velocity field. If we assume equal orders of differentiability, *i.e.*, if we choose $t_1 = \ldots = t_6 = 1$ then we must take $s_1 = \ldots = s_6 = 0$ and the principal part becomes

$$\mathcal{L}^p\,U = \begin{pmatrix} -\sqrt{2\nu}\frac{\partial u_1}{\partial x} \\ -\sqrt{2\nu}(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}) \\ -\sqrt{2\nu}\frac{\partial u_2}{\partial y} \\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \\ \sqrt{2\nu}(\frac{\partial T_1}{\partial x} + \frac{\partial T_2}{\partial y}) - \frac{\partial p}{\partial x} \\ \sqrt{2\nu}(\frac{\partial T_2}{\partial x} + \frac{\partial T_3}{\partial y}) - \frac{\partial p}{\partial y} \end{pmatrix}.$$

A simple calculation however, shows that $\det \mathcal{L}^p(\boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi}$, *i.e.*, the problem (6) is not elliptic in the sense of [1] *under the assumption of an equal differentiability.* The interpretation of this fact is that the velocity-pressure-stress system is not well posed if one assumes that all unknowns belong to $H^1(\Omega)$. A well-posed system will result if one assumes that $T_{ij}, p \in H^1(\Omega)$ and that $\mathbf{u} \in H^2(\Omega)^2$. This situation is quite different compared with the velocity-vorticity-pressure form of the Stokes equations. In the latter case there exist two different sets of indices such that the corresponding principal parts are uniformly elliptic and of the same total order four (in two-dimensions). The first set corresponds to different orders of differentiability for the vorticity and the velocity; the second set however, corresponds to the equal differentiability assumption; see [5].

Next we verify the complementing condition. Let $\boldsymbol{n}$ be the outer unit normal vector to Γ at some point P and let $\boldsymbol{\xi}$ be a unit tangent vector to Γ at the same point. Then

$$\det \mathcal{L}^p(\boldsymbol{\xi} + \tau\boldsymbol{n}) = \nu(1 + \tau^2)^2$$

and $M^+(\boldsymbol{\xi},\tau) = (\tau - i)^2$. Without loss of generality we may assume that the coordinate axes are aligned with the directions of $\boldsymbol{\xi}$ and $\mathbf{n}$ so that $\boldsymbol{\xi} = (1,0)$ and $\mathbf{n} = (0,-1)$. Then, (13) reduces to

$$\begin{aligned}
c_1\tau^2 - c_2\tau &= (\tau - i)^2 p_1(\tau)\\
c_1(\tau^3 - \tau) + c_2(\tau^2 - \tau) &= (\tau - i)^2 p_2(\tau)\\
c_1\nu(\tau^2 + 1) - c_2\nu\tau(\tau^2 + 1) &= (\tau - i)^2 p_3(\tau)\\
c_1\tau - c_2 &= (\tau - i)^2 p_4(\tau)
\end{aligned}$$

where c_i are constants and $p_i(\tau)$ are polynomials. Note that on the last line the right-hand side is at least a second degree polynomial, whereas the left-hand side is at most a first degree polynomial. Hence identity is possible if and only if $c_1 = c_2 = 0$, *i.e.* the complementing condition holds.

Let T, p and $\mathbf{u}$ be arbitrary smooth functions with p satisfying (8). Theorem 1 together with the uniqueness result from Proposition 1 implies that the following inequality holds for T, p and $\mathbf{u}$:

$$\|T\|_{q+1} + \|p\|_{q+1} + \|\mathbf{u}\|_{q+2} \le C\left(\|T - \sqrt{2\nu}\,\epsilon(\mathbf{u})\|_{q+1} + \|\nabla\cdot\mathbf{u}\|_{q+1} + \|\sqrt{2\nu}\,\nabla\cdot T - \nabla p\|_q\right). \quad (16)$$

The inequality (16) can be extended to functions $T \in H^{q+1}(\Omega)^3, p \in \tilde{H}^{q+1}(\Omega), \mathbf{u} \in H^{q+2}(\Omega)^2, \mathbf{u} = 0$ on Γ for $q \ge 0$ by a standard density argument. Note that if $F_1 \in H^{q+1}(\Omega)^3, f_2 \in \tilde{H}^{q+1}(\Omega)$ and $\mathbf{f}_3 \in H^q(\Omega)^2$ denote the right hand sides corresponding to the functions T, p and $\mathbf{u}$ then (16) can be stated as

$$\|T\|_{q+1} + \|p\|_{q+1} + \|\mathbf{u}\|_{q+2} \le C\left(\|F_1\|_{q+1} + \|f_2\|_{q+1} + \|\mathbf{f}_3\|_q\right).$$

For the analysis of the least-squares methods considered here we shall need to prove that (16) remains valid for negative values of the regularity index q. Results of this type are known to be true for elliptic systems of Petrovski type [26] but must be established for the system (6) which does not fall into this category. In our proof we use the same idea as in [26] of passing to the adjoint equation and the fact that (6) is a self-adjoint problem. The proof which is similar to the one in [5] is given here for the sake of completeness.

Theorem 2 *Let $U = (T, p, \mathbf{u}) \in D = \mathcal{D}(\bar{\Omega})^3 \times \tilde{\mathcal{D}}(\bar{\Omega}) \times \mathcal{D}(\bar{\Omega})^2$, $\mathbf{u} = 0$ on Γ and let F_1, f_2, and $\mathbf{f}_3$ be defined by* (6)*. Then, the* a priori *estimate* (16) *holds for all $q \in \mathbf{R}$.*

Proof. We introduce the product spaces

$$X_s = H^{s+1}(\Omega)^3 \times \tilde{H}^{s+1}(\Omega) \times [H^{s+2}(\Omega)]^2 \quad s \ge 0$$

$$Y_s = H^{s+1}(\Omega)^3 \times \tilde{H}^{s+1}(\Omega) \times [H^{s}(\Omega)]^2 \quad s \ge 0$$

together with their respective dual spaces

$$X_s^* = H^{-(s+1)}(\Omega)^3 \times \tilde{H}^{-(s+1)}(\Omega) \times [H^{-(s+2)}(\Omega)]^2 \quad s \ge 0$$

$$Y_s^* = H^{-(s+1)}(\Omega)^3 \times \tilde{H}^{-(s+1)}(\Omega) \times [H^{-s}(\Omega)]^2 \quad s \ge 0.$$

Let $\mathcal{L}$ denote the partial differential operator in (6). For $U \in D$ the estimate (16) holds for all $q \ge 0$ and can be written as

$$\|U\|_{X_q} \le C\|\mathcal{L}U\|_{Y_q}.$$

Note that because the problem defined by the operator $\mathcal{L} : X_s \mapsto Y_s$ together with the boundary condition $\mathbf{u} = 0$ is self-adjoint, the estimate (16) also holds for the solutions of the adjoint problem. We shall prove that

$$\|U\|_{Y_s^*} \leq C\|\mathcal{L}U\|_{X_s^*} \quad \forall U \in D;\; s \geq 0\,.$$

By the definition of the dual norm, uniqueness of the solutions to (6), (8) and by (16)

$$\|U\|_{Y_s^*} = \sup_{H\in D; H\neq 0} \frac{(U,H)}{\|H\|_{Y_s}} = \sup_{V\in D; V\neq 0} \frac{(U,\mathcal{L}V)}{\|\mathcal{L}V\|_{Y_s}} \leq C \sup_{V\in D; V\neq 0} \frac{(\mathcal{L}U,V)}{\|V\|_{X_s}} = C\|\mathcal{L}U\|_{X_s^*}$$

This establishes (16) for $q \leq -2$ and $q \geq 0$. For the intermediate values of q the result follows by application of the interpolation inequalities, see *e.g.* [25]. $\square$

4 Least-Squares Methods

In this section we define and analyse least-squares methods for the approximate solution of (6). For these methods to be optimal and practical it is critical to define properly the least-squares functional on which the least-squares minimization principle will be based.

These two issues, namely practicality and optimality of the resulting method, impose some contradicting demands on the formulation of the least-squares functionals. The method will be practical if it is possible to discretize it conformingly using simple continuous finite element spaces. Since the highest order of differentiation in the equations of (6) is one it is tempting to define the least-squares functional as the sum of the residuals of the equations in the L^2-norm. Unfortunately, this straightforward approach results in methods that are not optimal (see Section 5 for numerical examples). On the other hand, an optimal method can be derived from a least-squares functional in which each residual is measured in the norm of the Sobolev space determined by the Agmon-Douglis-Nirenberg index s_i of the respective equation along the same lines as demonstrated in [5]. For example, in the case of the system (6) one would have to measure the residuals of the continuity equation and of the three differential equations defining the components of the stress tensor in the H^1-norm because the indices of these equations are equal to -1. This approach, however, eliminates the advantages of the first order formulation, since now conforming discretizations are only possible with continuosly differentiable finite element spaces.

To resolve these contradicting issues we consider mesh dependent functionals in which the residual of each equation is measured in the L^2-norm multiplied by a suitable mesh-dependent weight. The purpose of these weights is to simulate norms in the stronger Sobolev spaces prescribed by the ADN a priori estimates. That is, we replace the norm $\|v^h\|_{-s_i}$ of the discrete function v^h in H^{-s_i} by the weighted norm $h^{s_i}\|v^h\|_0$ where h denotes some parameter of the discrete space. One can also deduce the appropriate weights by a length scale argument. Accordingly, the weighted least-squares functional for the velocity-pressure-stress equations (6) is defined as

$$\mathcal{J}^h(U) = \frac{1}{h^2}\|T - \sqrt{2\nu}\,\epsilon(\mathbf{u}) - F_1\|_0^2 + \frac{1}{h^2}\|\nabla\cdot\mathbf{u} - f_2\|_0^2 + \|\sqrt{2\nu}\,\nabla\cdot T - \nabla p - \mathbf{f}_3\|_0^2 \tag{17}$$

The weighted least-squares finite element method will be defined by considering minimization of

(17) over a suitable finite dimensional space $\mathbf{U}^h$ parametrized by h. We shall assume that

$$\mathbf{U}^h = S_1^3 \times S_1 \cap L_0^2(\Omega) \times (S_2 \cap H_0^1(\Omega))^2 \subset H^1(\Omega)^3 \times \tilde{H}^1(\Omega) \times H_0^1(\Omega)^2 \,. \tag{18}$$

We shall also assume that there exists a positive integer d such that the finite dimensional spaces S_j approximate optimally with respect to the spaces $H^{d+j}(\Omega)$, $j = 1, 2$. More precisely, we assume that for every $u \in H^{d+j}(\Omega)$ there exists an element $v^h \in S_j$ such that for $0 \leq r \leq 1$

$$\|u - v^h\|_r \leq C\, h^{d+j-r}\, \|u\|_{d+j} \,. \tag{19}$$

Finally we shall assume that the spaces S_j satisfy the inverse assumption, *i.e.*, that

$$\|v^h\|_1 \leq C\, h^{-1}\, \|v^h\|_0 \quad \forall v^h \in S_j \,. \tag{20}$$

Above assumptions are sufficiently general and hold for a large number of polynomial finite element spaces defined on uniformly regular triangulations, see [14].

Using standard techniques of the calculus of variations one can show, for any fixed value of h, that a minimizer of (17) out of the space $\mathbf{U}^h$ necessarily satisfies the variational problem (Euler-Lagrange equation)

$$\textit{find } U^h \in \mathbf{U}^h \textit{ such that } \mathcal{B}^h(U^h, V^h) = \mathcal{F}^h(V^h) \;\; \forall\, V^h \in \mathbf{U}^h \,, \tag{21}$$

where $U^h = (T^h, p^h, \mathbf{u}^h)$, $V^h = (S^h, q^h, \mathbf{v}^h)$ and

$$\begin{aligned} \mathcal{B}^h(U^h, V^h) &= \int_\Omega \frac{1}{h^2}(T^h - \sqrt{2\nu}\epsilon(\mathbf{u}^h)) : (S^h - \sqrt{2\nu}\epsilon(\mathbf{v}^h)) \\ &+ \frac{1}{h^2}(\nabla\cdot\mathbf{u}^h)(\nabla\cdot\mathbf{v}^h) + (\sqrt{2\nu}\nabla\cdot T^h - \nabla p^h)\cdot(\sqrt{2\nu}\nabla\cdot S^h - \nabla q^h)\, dx \\ \mathcal{F}^h(V^h) &= \int_\Omega \frac{1}{h^2} F_1 : (S^h - \sqrt{2\nu}\epsilon(\mathbf{v}^h)) + \frac{1}{h^2} f_2 \nabla\cdot\mathbf{v}^h + \mathbf{f}_3\cdot(\sqrt{2\nu}\nabla\cdot S^h - \nabla q^h) dx \,. \end{aligned} \tag{22}$$

The weighted least-squares finite element method is now completely defined by the variational problem (21). For the success of the method it is important to establish that (17) has a unique minimizer out of the space $\mathbf{U}^h$.

Theorem 3 *The least-squares functional (17) has unique minimizer out of the space (18) for any $h < 1$.*

Proof. We shall establish that the bilinear form $\mathcal{B}^h(\cdot,\cdot)$ is continuous and coercive on $\mathbf{U}^h \times \mathbf{U}^h$ when $\mathbf{U}^h$ is equiped with the norm of $\mathbf{X} = L^2(\Omega)^3 \times L_0^2(\Omega) \times H_0^1(\Omega)^2$. Then, from the Lax-Milgram Lemma it will follow that the variational problem (21) has a unique solution in $\mathbf{U}^h$ and since (21) is the necessary condition for (17) the Theorem will be proved.

Let $U^h, V^h \in \mathbf{U}^h$. We use the a priori estimate (16) with $q = -1$ and the continuity of the imbedding $L^2(\Omega) \subset H^{-1}(\Omega)$ to find that

$$\begin{aligned} \|T^h\|_0^2 + \|p^h\|_0^2 + \|\mathbf{u}^h\|_1^2 &\leq C\left(\|T^h - \sqrt{2\nu}\,\epsilon(\mathbf{u}^h)\|_0^2 + \|\nabla\cdot\mathbf{u}^h\|_0^2 + \|\sqrt{2\nu}\,\nabla\cdot T^h - \nabla p^h\|_{-1}^2\right) \\ &\leq C\left(\frac{1}{h^2}\|T^h - \sqrt{2\nu}\,\epsilon(\mathbf{u}^h)\|_0^2 + \frac{1}{h^2}\|\nabla\cdot\mathbf{u}^h\|_0^2 + \|\sqrt{2\nu}\,\nabla\cdot T^h - \nabla p^h\|_0^2\right) \\ &= C\,\mathcal{B}^h(U^h, U^h) \,. \end{aligned}$$

To estimate $\mathcal{B}^h(U^h, V^h)$ from above in the norm of $\mathbf{X}$ we use the inverse inequality (20). Then the result easily follows.$\Box$

Because the factor h^{-2} will appear in the continuity estimate for $\mathcal{B}^h(\cdot,\cdot)$, Theorem 3 cannot be used to derive opitmal error bounds. However, this does not mean that the weights destroy the convergence rates; on the contrary, the more careful error analysis of the next section will reveal that these weights are essential for the optimal convergence of the least-squares approximations. In contrast, application of the least-squares principle with the velocity-vorticity-pressure equations leads to methods which, in order to achieve optimal rates, may or may not require weights in the functional, see [5]. In the latter case the necessity of the weights or lack of it is essentialy determined by the type of the boundary conditions considered with the differential operator. For some boundary conditions, like the velocity boundary condition, the complementing condition holds only under the assumption of a different order of differentiability and as a consequence, the weights are necessary in the functional. For other boundary conditions, like the normal velocity-pressure boundary condition, the complementing condition is satisfied with all equation indices assumed equal to zero. In such a case one can consider functionals without the weights. For the velocity-pressure-stress equations (6) the possibility for zero equation indices is completely ruled out by the fact that the differential operator in (6) is uniformly elliptic only under the assumption of a different order of differentiability, *i.e.*, when some of the equation indices are strictly negative.

4.1 Error Estimates

In this section we derive the error estimates for the approximations U^h under the assumption that the solution $U = (T, p, \mathbf{u})$ of (6) belongs to $\mathbf{U} = H^{l+1}(\Omega)^3 \times \tilde{H}^{l+1} \times (H^{l+2}(\Omega) \cap H_0^1(\Omega))^2$ for some integer $l \geq 0$. We shall also assume that the space $\mathbf{U}^h$ is defined by (18) and that the spaces S_j satisfy (19) for some integer $d > 0$. The necessity of this requirement will become clear from the proof of Lemma 2 below. By $\mathcal{L}_{ij}$ we shall denote the components of the differential operator $\mathcal{L}$ in (15) corresponding to the system (6). We first derive an estimate of continuity type for the errors $U - U^h$.

Lemma 1 *Let $U = (T, p, \mathbf{u}) \in \mathbf{U}$ be arbitrary functions, let F_1, f_2 and $\mathbf{f}_3$ be defined by* (6) *and let $U^h = (T^h, p^h, \mathbf{u}^h) \in \mathbf{U}^h$ denote the solution of the variational problem* (21). *Then*

$$\mathcal{B}^h(U - U^h, U - U^h)^{\frac{1}{2}} \leq C h^{\tilde{d}} \left(\|T\|_{\tilde{d}+1} + \|p\|_{\tilde{d}+1} + \|\mathbf{u}\|_{\tilde{d}+2} \right) \tag{23}$$

where $\tilde{d} = \min\{l, d\}$.

Proof. Let $V^h = (S^h, q^h, \mathbf{v}^h) \in \mathbf{U}^h$. The error $U - U^h$ satisfies the usual orthogonality relation $\mathcal{B}^h(U - U^h, V^h) = 0$ and therefore

$$\mathcal{B}^h(U - U^h, U - U^h)^{\frac{1}{2}} \leq \mathcal{B}^h(U - V^h, U - V^h)^{\frac{1}{2}} \quad \forall V^h \in \mathbf{U}^h \,.$$

Using the approximation properties (19) of the spaces S_j we find that

$$\inf_{V^h \in \mathbf{U}^h} \mathcal{B}^h(U - V^h, U - V^h)^{\frac{1}{2}} =$$

$$
\begin{aligned}
&= \inf_{V^h \in \mathbf{U}^h} \Big(h^{-2} \, \Big(\| (T - S^h) - \sqrt{2\nu}\epsilon(\mathbf{u} - \mathbf{v}^h) \|_0^2 + \| \nabla \cdot (\mathbf{u} - \mathbf{v}^h) \|_0^2 \Big) + \| \sqrt{2\nu} \nabla \cdot (T - S^h) - \nabla (p - q^h) \|_0^2 \Big)^{\frac{1}{2}} \\
&\leq C \inf_{V^h \in \mathbf{U}^h} \Big(h^{-1} \| T - S^h \|_0 + \| T - S^h \|_1 + h^{-1} \| \mathbf{u} - \mathbf{v}^h \|_1 + \| p - q^h \|_1 \Big) \\
&\leq C h^{\tilde{d}} \Big(\| T \|_{\tilde{d}+1} + \| p \|_{\tilde{d}+1} + \| \mathbf{u} \|_{\tilde{d}+2} \Big) . \quad \square
\end{aligned}
$$

Our next Lemma establishes the stability of the form $\mathcal{B}^h$.

Lemma 2 *Assume that the spaces S_j satisfy* (19) *for some integer $d \geq 1$. Let q satisfy*

$$
1 \leq -q \leq d \, .
$$

Then, for U and U^h as in Lemma 1,

$$
\| T - T^h \|_{q+1} + \| p - p^h \|_{q+1} + \| \mathbf{u} - \mathbf{u}^h \|_{q+2} \leq C \, h^{-q} \, \mathcal{B}^h (U - U^h, U - U^h)^{\frac{1}{2}} \, . \tag{24}
$$

Proof. In the proof of this lemma we use some ideas of [2]. We also recall the definition of the weights s_i and t_j from Section 3. Application of (16) with $q \leq -1$ to the error $U - U^h$ yields the estimate

$$
\begin{aligned}
& C_1 \Big(\| T - T^h \|_{q+1}^2 + \| p - p^h \|_{q+1}^2 + \| \mathbf{u} - \mathbf{u}^h \|_{q+2}^2 \Big) \leq \\
& \qquad \leq \| (T - T^h) - \sqrt{2\nu} \, \epsilon(\mathbf{u} - \mathbf{u}^h) \|_{q+1} + \| \nabla \cdot (\mathbf{u} - \mathbf{u}^h) \|_{q+1} + \| \sqrt{2\nu} \, \nabla \cdot (T - T^h) - \nabla (p - p^h) \|_q \, .
\end{aligned}
$$

The terms on the right-hand side above are of the form $\| \sum_j \mathcal{L}_{ij} (U_j - U_j^h) \|_{q - s_i}$ and the estimate (24) will follow if each one of these terms can be estimated by a constant times $h^{-q} \mathcal{B}^h (U - U^h, U - U^h)$. This can be accomplished by interpolation between the spaces $H^{s_i - d}(\Omega)$ and $L^2(\Omega)$ if $s_i - d \leq q - s_i \leq 0$, thus the assumption $d \geq 1$. According to (9),

$$
\| \sum_j \mathcal{L}_{ij} (U_j - U_j^h) \|_{s_i - d} = \sup_{f_i \in \mathcal{D}} \frac{\Big(\sum_j \mathcal{L}_{ij} (U_j - U_j^h), f_i \Big)}{\| f_i \|_{d - s_i}} \, .
$$

where $\mathcal{D}$ is a space of smooth functions which is dense in $H^{d-s_i}(\Omega)$ or $\tilde{H}^{d-s_i}(\Omega)$. Let $F_1 \in \mathcal{D}(\bar{\Omega})^3$, $f_2 \in \tilde{\mathcal{D}}(\bar{\Omega})$, and $\mathbf{f}_3 \in \mathcal{D}(\bar{\Omega})^2$. The above space for f_2 can be choosen thanks to our assumption that the boundary conditions are satisfied exactly. Indeed, in this case $\sum_j \mathcal{L}_{4j} (U_j - U_j^h) = \nabla \cdot (\mathbf{u} - \mathbf{u}^h)$ has zero mean and the supremum in the corresponding dual norm has to be taken with respect to $\tilde{\mathcal{D}}(\bar{\Omega})$. Therefore, the function f_2 meets the solvability condition (7) for the problem:

$$
\begin{aligned}
S - \sqrt{2\nu} \, \epsilon(\mathbf{v}) &= F_1 \ \text{ in } \Omega \\
\nabla \cdot \mathbf{v} &= f_2 \ \text{ in } \Omega \\
\sqrt{2\nu} \, \nabla \cdot S - \nabla q &= \mathbf{f}_3 \ \text{ in } \Omega \\
\mathbf{v} &= 0 \ \text{ on } \Gamma
\end{aligned} \tag{25}
$$

and for every smooth right-hand side in the indicated spaces this problem will have unique solution. If the boundary conditions were not exactly satisfied, one would have to consider an arbitrary smooth function f_2. Then the system (25) must be modified (see [2], [28]) in order to guarantee its solvability.

Now, let $V = (S, q, \mathbf{v})$ be the solution of (25) where only one of the right hand-side functions is taken to be different from zero. Let us suppose that this function is f_i and let V^h denote the least-squares approximation to V computed by (21). Because of the smoothness of the functions S, q and $\mathbf{v}$ the estimate (23) will hold with $\tilde{d} = d$. Then, we use the orthogonality of the error, definition (22), and the estimates (23) and (16) to find an upper bound for the term $\Big(\sum_j \mathcal{L}_{ij}(U_j - U_j^h)\,,\, f_i\Big)$:

$$
\begin{aligned}
\Big(\sum_j \mathcal{L}_{ij}(U_j - U_j^h)\,,\, f_i\Big) &= h^{-2s_i}\Big(h^{2s_i}\sum_j \mathcal{L}_{ij}(U_j - U_j^h)\,,\, f_i\Big) \\
&= h^{-2s_i}\sum_k \Big(h^{2s_k}\sum_j \mathcal{L}_{kj}(U_j - U_j^h)\,,\, f_k\Big) \\
&= h^{-2s_i}\sum_k \Big(h^{2s_k}\sum_j \mathcal{L}_{kj}(U_j - U_j^h)\,,\, \sum_m \mathcal{L}_{km}V_m\Big) \\
&= h^{-2s_i}\mathcal{B}^h(U - U^h, V) = h^{-2s_i}\mathcal{B}^h(U - U^h, V - V^h) \\
&\leq Ch^{-2s_i}(\mathcal{B}^h(U - U^h, U - U^h))^{1/2}(\mathcal{B}^h(V - V^h, V - V^h))^{1/2} \\
&\leq Ch^{d-2s_i}(\mathcal{B}^h(U - U^h, U - U^h))^{1/2}\Big(\|S\|_{d+1} + \|q\|_{d+1} + \|\mathbf{v}\|_{d+2}\Big) \\
&\leq Ch^{d-2s_i}(\mathcal{B}^h(U - U^h, U - U^h))^{1/2}\|f_i\|_{d-s_i}\,.
\end{aligned}
$$

Therefore,

$$\|\sum_j \mathcal{L}_{ij}(U_j - U_j^h)\|_{s_i-d} \leq Ch^{d-2s_i}(\mathcal{B}^h(U - U^h, U - U^h))^{1/2}\,.$$

For the estimate of $\sum_j \mathcal{L}_{ij}(U_j - U_j^h)$ in the norm of $L^2(\Omega)$ we use the definition of the form $\mathcal{B}^h(\cdot,\cdot)$ to find

$$\|\sum_j \mathcal{L}_{ij}(U_j - U_j^h)\|_0 \leq Ch^{-s_i}(\mathcal{B}^h(U - U^h, U - U^h))^{1/2}\,.$$

Now, the estimate for the $(q - s_i)$-th norm can be found by interpolation between $H^{s_i-d}(\Omega)$ and $H^0(\Omega)$. For d and q choosen according to the statement of the lemma

$$s_i - d \leq q - s_i \leq 0$$

for all equation indices s_i and if

$$\theta = \frac{s_i - q}{d - s_i}$$

then the space $H^{q-s_i}(\Omega)$ can be defined by interpolation (see [25]):

$$\Big[H^0(\Omega), H^{s_i-d}(\Omega)\Big]_\theta = H^{q-s_i}(\Omega)$$

The application of the interpolation inequality [25] yields

$$
\begin{aligned}
&\|\sum_j \mathcal{L}_{ij}(U_j - U_j^h)\|_{q-s_i} \leq C\|\sum_j \mathcal{L}_{ij}(U_j - U_j^h)\|_{s_i-d}^{\theta}\|\sum_j \mathcal{L}_{ij}(U_j - U_j^h)\|_0^{1-\theta} \\
&\leq \; Ch^{(d-2s_i)\theta}h^{-s_i(1-\theta)}(\mathcal{B}^h(U - U^h, U - U^h))^{1/2} = Ch^{-q}(\mathcal{B}^h(U - U^h, U - U^h))^{1/2}\,.
\end{aligned}
$$

Then (24) easily follows. $\square$

We are now prepared to prove the main error estimate.

Theorem 4 *Let $U \in \mathbf{U}$ solve the problem (6)-(8). Let q and d be defined as in Lemma 2 and let $\tilde{d} = \min\{d, l\}$. Then, the least-squares approximation $U^h \in \mathbf{U}^h$ of U satisfies*

$$\|T - T^h\|_{q+1} + \|p - p^h\|_{q+1} + \|\mathbf{u} - \mathbf{u}^h\|_{q+2} \le C\, h^{\tilde{d}-q}\Big(\|T\|_{\tilde{d}+1} + \|p\|_{\tilde{d}+1} + \|\mathbf{u}\|_{\tilde{d}+2}\Big)\,. \qquad (26)$$

Proof. Using (23) and (24) it follows that

$$\begin{aligned}
\|T - T^h\|_{q+1} + \|p - p^h\|_{q+1} + \|\mathbf{u} - \mathbf{u}^h\|_{q+2} &\le C\, h^{-q}\, \mathcal{B}^h(U - U^h, U - U^h)^{\frac{1}{2}} \\
&\le C\, h^{-q} \inf_{V_h \in \mathbf{U}^h} \mathcal{B}^h(U - V^h, U - V^h)^{\frac{1}{2}} \\
&\le C\, h^{\tilde{d}-q}\Big(\|T\|_{\tilde{d}+1} + \|p\|_{\tilde{d}+1} + \|\mathbf{u}\|_{\tilde{d}+2}\Big)\,. \quad \square
\end{aligned}$$

A few comments are now in order. For a given $d \ge 1$ the error estimate in Theorem 4 will be optimal if $l \ge d$, *i.e.*, if the solution of the problem (6)-(8) is sufficiently regular. Let us assume that $d = 1$, *i.e.*, that the finite element spaces S_1 and S_2 approximate optimally with respect to $H^2(\Omega)$ and $H^3(\Omega)$, accordingly. Then, $q = -1$, and the estimate (26) is optimally accurate if $l \ge 1$ in (18). For example, if $T \in H^2(\Omega)^3$, $p \in H^2(\Omega)$ and $\mathbf{u} \in H^3(\Omega)^2$, *i.e.*, if $l = 1$ in (18), then one can get the optimal error estimate

$$\|T - T^h\|_0 + \|p - p^h\|_0 + \|\mathbf{u} - \mathbf{u}^h\|_1 \le C\, h^2\Big(\|T\|_2 + \|p\|_2 + \|\mathbf{u}\|_3\Big)$$

using piecewise linear approximation for the stresses and the pressure and piecewise quadratic approximation for the velocity.

Note that because the space S_2 must approximate optimally with respect to $H^3(\Omega)$ our analysis does not cover the case of piecewise linear approximations for the velocity. Thus, it is a legitimate question to ask what happens when piecewise linear approximations are used for all six unknowns. Results of some computational experiments, that will be discussed in the next section, suggest that such an implementation of the weighted least-squares method (21) is not optimal, *i.e.*, it does not achieve the theoretical rates of h^2 and h^1 for the errors in the L^2 and H^1-norms respectively. Thus, we may conjecture that the approximation properties of the discrete spaces demanded by our error analysis are not redundant or artificial. If, on the other hand, one uses piecewise quadratic elements for all unknowns then all approximability assumptions of Theorem 4 are met and the error estimate (26) is valid. However, this estimate will not be optimal when the pressure and the stress components are approximated by piecewise quadratic finite element spaces.

The choice of the indices s_i in Section 3 implies that in Lemma 2 and in Theorem 4 the index q cannot be greater then -1. Thus, for the approximations of the stress and the pressure Theorem 4 provides only L^2-norm estimates of the error. If the approximation spaces S_j satisfy the inverse assumption (20) then one can obtain error estimates in the stronger H^1-norm.

Corollary 1 *Assume that (20) holds for the spaces S_1 and S_2. Then,*

$$\|T - T^h\|_{1,\Omega} \leq Ch^d \left(\|T\|_{d+1} + \|p\|_{d+1} + \|\mathbf{u}\|_{d+2}\right) \tag{27}$$

$$\|p - p^h\|_{1,\Omega} \leq Ch^d \left(\|T\|_{d+1} + \|p\|_{d+1} + \|\mathbf{u}\|_{d+2}\right) . \tag{28}$$

Proof. Using the approximation properties (19), the estimate (26) with $q = -1$ and the inequality (20) we find that

$$\begin{aligned}
\|T - T^h\|_{1,\Omega} &\leq \|T - S^h\|_{1,\Omega} + \|S^h - T^h\|_{1,\Omega} \\
&\leq C\left(h^d\|T\|_{d+1} + h^{-1}\|T^h - S^h\|_{0,\Omega}\right) \\
&\leq C\left(h^d\|T\|_{d+1} + h^{-1}\left(\|T - S^h\|_{0,\Omega} + \|T - T^h\|_{0,\Omega}\right)\right) \\
&\leq C\left(h^d\|T\|_{d+1} + h^{d-q-1}\left(\|T\|_{d+1} + \|p\|_{d+1} + \|\mathbf{u}\|_{d+2}\right)\right) \\
&\leq Ch^d\left(\|T\|_{d+1} + \|p\|_{d+1} + \|\mathbf{u}\|_{d+2}\right) .
\end{aligned}$$

The estimate (28) is derived in an identical manner. □

4.2 Condition Numbers

In this section we derive upper bounds for the condition numbers of the discretization matrix of the weighted least-squares method (21). Recall that the spectral condition number for a $n \times n$ invertible matrix A is defined as

$$\mathrm{cond}(A) = \|A\|_2\|A^{-1}\|_2 = \frac{\lambda_{max}}{\lambda_{min}} . \tag{29}$$

The estimates of the condition numbers will be made under the assumption that the spaces S_j satisfy the following property:

> if $\{\phi_i\}$ is a basis of S_j then, for every $u^h = \sum \phi_i u_i^h \in S_j$ there exist positive constants α and β such that
> $$\alpha h^N |u^h|^2 \leq \|u^h\|_0 \leq \beta h^N |u^h|^2 , \tag{30}$$
> where $|u^h|^2 = \sum u_i^2$ and N denotes the space dimension.

For example, (30) holds if the triangulation $\mathcal{T}_h$ is regular and $\{\phi_i\}$ is the nodal basis; see [14].

Lemma 3 *Let A denote the discretization matrix of the weighted least-squares method (21) for the Stokes problem (6). Then*

$$\mathit{cond}(A) \leq Ch^{-4} . \tag{31}$$

Proof. Let a denote the vector of the coefficients of the finite element function $U^h \in \mathbf{U}^h$ with respect to a suitable basis. For simplicity, let u, T and p denote the coefficients of $\mathbf{u}^h$, T^h and p^h, respectively. Then

$$a^T A a = \mathcal{B}^h(U^h, U^h)$$

and

$$|a|^2 = |u|^2 + |T|^2 + |p|^2 \,.$$

From (22) it follows that

$$\mathcal{B}^h(U^h, U^h) = \frac{1}{h^2}\|T^h - \sqrt{2\nu}\,\epsilon(\mathbf{u}^h)\|_0^2 + \frac{1}{h^2}\|\nabla\cdot\mathbf{u}^h\|_0^2 + \|\sqrt{2\nu}\,\nabla\cdot T^h - \nabla p^h\|_0^2$$

As a result, for the upper bound we find

$$\begin{aligned}
\mathcal{B}^h(U^h, U^h) &\leq C\left(h^{-2}\|T^h\|_0^2 + \|T^h\|_1^2 + \|p^h\|_1^2 + h^{-2}\|\mathbf{u}^h\|_1^2\right) \\
&\leq C\frac{1}{h^4}\left(\|T^h\|_0^2 + \|p^h\|_0^2 + \|\mathbf{u}^h\|_0^2\right) \\
&\leq C_2 h^{N-4}\left(|T|^2 + |p|^2 + |u|^2\right) \,.
\end{aligned}$$

For the lower bound we use the ADN a apriori estimate (16) with $q = -1$ and the continuity of the imbedding $L^2(\Omega) \subset H^{-1}(\Omega)$

$$\begin{aligned}
\mathcal{B}^h(U^h, U^h) &\geq C\left(\|T^h - \sqrt{2\nu}\,\epsilon(\mathbf{u}^h)\|_0 + \|\nabla\cdot\mathbf{u}^h\|_0 + \|\sqrt{2\nu}\,\nabla\cdot T^h - \nabla p^h\|_0\right) \\
&\geq C\left(\|T^h - \sqrt{2\nu}\,\epsilon(\mathbf{u}^h)\|_0 + \|\nabla\cdot\mathbf{u}^h\|_0 + \|\sqrt{2\nu}\,\nabla\cdot T^h - \nabla p^h\|_{-1}\right) \\
&\geq C\left(\|T^h\|_0^2 + \|p^h\|_0^2 + \|\mathbf{u}^h\|_1^2\right) \\
&\geq C\left(\|T^h\|_0^2 + \|p^h\|_0^2 + \|\mathbf{u}^h\|_0^2\right) \\
&\geq C_1 h^N\left(|T|^2 + |p|^2 + |u|^2\right) \,.
\end{aligned}$$

Now (31) easily follows. $\Box$

It is of interest to know whether the upper bound (31) is sharp or not. At present we do not have a definite answer for this question, but some computational results presented in the next section suggest that the above estimate might be too conservative.

5 Numerical Results

In this section we present some results from numerical experiments with two different implementations of the weighted least-squares method (21). The first implementation uses piecewise linear finite elements for the approximation of the stress components and the pressure and piecewise quadratic finite elements for the approximation of the velocity. In the second implementation we use piecewise linear finite elements for all unknowns. For brevity we shall refer to these implementations as the *linear-quadratic* and *linear-linear*, respectively. In all numerical examples the computational domain is taken to be the unit square and we employ a uniform triangulation. We begin with a numerical study of the convergence rates. The main goal of this study is to show that the weights in the least-squares functional (17) are necessary for the optimality of the method. Then we discuss a computational study of the condition numbers of the corresponding discretization matrices.

5.1 Numerical study of convergence rates

We take for our domain the unit square $\Omega = \{0 \le x \le 1, 0 \le y \le 1\}$ and consider the generalized velocity-pressure-stress Stokes equations (6). We will define the data functions F_1, f_2, and $\mathbf{f}_3$ by choosing an exact solution $U = (T, p, \mathbf{u})$ and then substituting this solution into the equations in (6). We choose the following exact solution:

$$\begin{aligned} u_1 = u_2 &= \sin(\pi x)\sin(\pi y) \\ T_1 = T_2 = T_3 &= \sin(\pi x)\exp(\pi y) \\ p &= \cos(\pi x)\exp(\pi y)\,. \end{aligned} \tag{32}$$

The exact solution (32) is smooth and satisfies the homogeneous velocity boundary condition.

The goal of the first numerical experiment is to confirm that the weights are necessary for the optimal convergence rates. For this purpose we consider the linear-quadratic implementation because it fits into the error analysis of Section 4.1. First, we compute the approximations T^h, p^h and $\mathbf{u}^h$ for the exact solution (32) using the weighted least-squares method (21). Theoretically, (26), (27) and (28) should hold for T^h, p^h and $\mathbf{u}^h$, *i.e.*, we expect the following rates

$$\|T - T^h\|_r = O(h^{2-r}) \quad \text{and} \quad \|p - p^h\|_r = O(h^{2-r}) \quad \text{for} \ \ r = 0, 1\,,$$

and

$$\|\mathbf{u} - \mathbf{u}^h\|_r = O(h^{3-r}) \quad \text{for} \ \ r = 1\,.$$

Then we remove the weights from the functional (17) and perform the calculations for the same grid sizes and the same exact solution and compare the convergence rates for both cases. The log-log plots of the corresponding L^2 and H^1 errors vs. the number of grid intervals in each direction are presented on Figures 1 and 2 respectively. The solid lines in these figures correspond to the results obtained with the weighted least-squares functional (17). The dashed lines correspond to the results computed without the weights in (17). (Note that in the figures, $u = u_1$ and $v = u_2$.) The slopes of the various curves in Figures 1 - 2 are proportional to the numerical convergence rates. Comparing the solid and the dashed lines on these figures we can conclude that the approximations computed with the weights converge at a faster rate than the approximations computed without the weights. The above conclusions can be quantized by computing the slope of a least squares straight-line fit to the various curves in the figures. The results are given in Table 1. In Table 1 the column WLS corrsponds to the convergence rates obtained with the weighted least-squares functional (17), the column LS corresponds to the convergence rates obtained without the weights and, finally, the column BA gives the rate of the best approximation out of the finite element space. We note that, as it is usually the case, the computed L^2 rates are less reliable then their H^1 counterparts. The rates in WLS columns are in agreement with the theoretical error estimates. For the pressure and the stress approximations the H^1 rates in columns WLS and LS are almost identical. However, for the velocity approximations the H^1 rates in column LS are approximately of one order less then the H^1 rates in column WLS. As a result, we can conclude that without the weights the least-squares method looses one order of accuracy.

Table 1: Convergence rates with and without the weights

Rates	L^2 error			H^1 error		
Example 32	WLS	LS	BA	WLS	LS	BA
u	3.59	1.11	3.00	2.85	1.00	2.00
v	3.13	1.28	3.00	2.77	1.17	2.00
T_{11}	2.42	1.25	2.00	0.99	0.94	1.00
T_{12}	2.48	1.14	2.00	1.01	0.99	1.00
T_{22}	2.34	1.26	2.00	1.05	0.76	1.00
p	2.40	0.94	2.00	1.10	0.92	1.00

Table 2: Convergence rates for linear-quadratic and linear-linear weighted least-squares

Rates	L^2 error			H^1 error		
Example 32	LL	LQ	BA	LL	LQ	BA
u	1.91	N/A	2.00	1.00	N/A	1.00
v	2.01	N/A	2.00	1.02	N/A	1.00
T_{11}	1.42	2.42	2.00	0.77	0.99	1.00
T_{12}	1.50	2.48	2.00	0.97	1.01	1.00
T_{22}	1.60	2.34	2.00	0.84	1.05	1.00
p	0.42	2.40	2.00	0.58	1.10	1.00

In the next numerical experiment we compare the linear-quadratic implementation of the weighted least-squares method and the linear-linear implementation of the same method. Recall that the error analysis of Section 4.1 does not cover the case of piecewise linear approximation of the velocity. If such an implementation results in an optimally accurate method, then we would expect to observe the following convergence rates

$$\|T - T^h\|_r = O(h^{2-r}) \quad \text{and} \quad \|p - p^h\|_r = O(h^{2-r}) \quad \text{for} \quad r = 0, 1\,,$$

and

$$\|\mathbf{u} - \mathbf{u}^h\|_r = O(h^{2-r}) \quad \text{for} \quad r = 0, 1\,.$$

We perform computations with both versions of (21) using the same uniform grids. The log-log plots of the L^2 and H^1 errors are presented on Figures 3 and 4 respectively. The solid line now corresponds to the computations with the linear-quadratic implementation and the dashed line is for the results of the linear-linear implementation. In Table 2 we give the slopes of the straight-line least-squares fit to the various curves on Figures 3 and 4. Columns LL contain convergence rates for the linear-linear approximation and columns LQ contain convergence rates for the linear-quadratic approximation. Note that the rates for the L^2 and H^1 errors of the velocity in columns LL are in good agreement with the optimal theoretical rates of 2 and 1 respectively. However, the rates for the stresses and the pressure are suboptimal compared to the rates in column BA. Moreover, these rates are worse than the rates for the same variables in columns LQ (we do not compare the LL and LQ columns for the velocity because of the different approximation spaces used for this unknown). As a result, we can conclude that, at least computationally, the linear-linear method is not optimal.

5.2 Numerical study of conditioning

The main objectives of the numerical experiments in this section are to investigate the influence of the weights and the type of the finite element spaces used in the least-squares method on the conditioning of the discretization matrices. For this purpose we implement the methods with assembly of the discretization matrix. Then condition numbers are estimated using the DPBCO routine from the LINPACK subroutine library.

For the first task we consider the linear-quadratic implementation of (21). Then we use the same finite element spaces but remove the weights. Results are presented on Figures 5 and 6. The solid line on Figure 5 is for the condition number of the weighted method and the dashed line is for the condition number of the method without the weights. Since the lines on Figure 5 are almost parallel one can conjecture that, at least numerically, *addition of the weights does not influence significantly the rate of growth of the condition numbers*. Indeed, the ratio of these condition numbers is very close to two; see Figure 6. Also, from Figure 5 one can infer that the condition number in both cases is of $O(h^{-2})$ so that the estimate (31) is seemingly very pessimistic.

Finally, on Figure 7 we compare the condition numbers for the linear-linear implementatation of the weighted least-squares method (21) (dashed line) with the condition numbers for the linear-quadratic impelementation of the same method (solid lines). Again, the slopes of these lines are almost parallel and one can conclude that *the choice of the discretization space does not affect significantly the rate of growth of the condition numbers*. Moreover, the ratio of these condition numbers is very close to one; see Figure 8, *i.e.*, conditioning of the method is less affected by the choice of the finite element spaces then by the addition of the weights.

6 Concluding Remarks

We have formulated and analyzed a least-squares finite element method for the approximate solution of the Stokes equtions based on a velocity-pressure-stress form of these equations. This method has convergence properties similar to the least-squares method based on velocity-vorticity-pressure Stokes equations [5]. Although the method based on the velocity-pressure-stress equations has more unknowns it is suitable when a direct approximation of the extra stress tensor is desired. The method described above extends without any difficulties to the three-dimensional case. The number of the unknowns increases to ten, the corresponding system has a total order of six and hence the velocity boundary condition need not be augmented. As in two-dimensions, for the ellipticity of the velocity-pressure-stress equations in three-dimensions we must assume negative indices for the continuity equation and the six differential equations that define the stress tensor. Definition of the least-squares functional in this case is also obvious and the proofs of error estimates are identical to the ones in Section 4.1. It is worth noting that in the case of the velocity-vorticity-pressure system formulation of least-squares methods in three-space dimensions is more elaborate. For example, in three-dimensions this system has seven equations and seven unknowns and thus it cannot be elliptic in the sense of the definitions in Section 3. To fix this one adds a seemingly redundant equation, involving the vorticity, and a slack variable [11]. Then the total order of the system increases to eight and the velocity boundary condition (or any other boundary condition taken from the Stokes problem in primitive variables) must be augmented with an additional condition; see [4] and [5].

The computational examples of Section 5 illustrate the fact that the attractive theoretical properties of the least-squares approach can be succesfuly implemented and that the resulting numerical methods are robust and efficient. In particular, our experiments indicate that all unknowns, including the stress tensor, are approximated optimally and that the weights in (17) are essential for the optimality of the computational results.

In conclusion, let us mention some of the issues that require further investigation. The first issue is the numerically suboptimal behaviour of the linear-linear implementation of the least-squares method. Although the case of the linear approximation for the velocity is not covered by the theory of Section 4.1 this does not automatically rule out the possibility that such an implementation might be optimal. On the other hand the numerical evidence suggests that the use of piecewise linear finite elements for all unknowns may not result in an optimlly accurate method.

Another open issue concerns the conditioning of the discretization matrices. Our numerical study of the condition numbers reveals the somewhat surprising fact that the rate of growth of these condition numbers is not affected significantly by the weights in (21). This suggests that the upper bound (31) might be overly pessimistic. This phenomena is not limited to the least-squares methods based on the velocity-pressure-stress Stokes equations. In [4] we observed similar behavior of the conditioning for the weighted least-squares method based on the velocity-vorticity-pressure Stokes equations.

Acknowledgement Authors are grateful to the reviewer for the careful, thorough and thoughtful comments and suggestions that helped to improve significantly this paper.

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References

[1] S. Agmon, A. Douglis, and L. Nirenberg; Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, *Comm. Pure Appl. Math.*, **17**, 1964, 35-92.

[2] A. Aziz, R. Kellogg, and A. Stephens; Least-squares methods for elliptic systems, *Math. of Comp.*, **44**,169, 1985, 53-70.

[3] M. A. Behr, L. P. Franca, T. E. Tezduyar; Stabilized finite element methods for the velocity-pressure-stress formulation of incompressible flows, *Computer Methods in Appl. Mech. Engn.*, **104**, 1993, 31-48.

[4] P. Bochev; *Least-Squares Methods for the Stokes and Navier-Stokes Equations*, Ph.D Thesis, Virginia Polytechnic Institute and State University, Blacksburg, 1994.

[5] P. Bochev and M. Gunzburger; Analysis of least-squares finite element methods for the Stokes equations, to appear in *Math. Comp.*

[6] P. Bochev and M. Gunzburger; Analysis of weighted least-squares finite element method for the Navier-Stokes equations, Proc. IMACS 14th World Congress, Georgie Tech, Atlanta, 1994.

[7] P. Bochev and M. Gunzburger; Accuracy of least-squares methods for the Navier-Stokes equations, *Comput. & Fluids*, **22** 4/5, 1993, 549-563

[8] J. H. Bramble, A. H. Schatz; Least-squares methods for $2m$th order elliptic boundary value problems, *Math. Comp.* **25**, 113, 1971, 1-32.

[9] C.-L. Chang; A mixed finite element method for the Stokes problem: an acceleration-presure formulation, *Appl. Math. Comput.* **36**, 1990, 135-146.

[10] C.-L. Chang; Least-squares finite-element method for incompressible flow in 3-D, to appear.

[11] C.-L. Chang and M. Gunzburger; A finite element method for first order systems in three dimensions, *Appl. Math. Comput.* **23**, 1987, 171-184.

[12] C.-L. Chang and B.-N. Jiang; An error analysis of least-squares finite element methods of velocity-vorticity-pressure formulation for the Stokes problem, *Comp. Meth. Appl Mech. Engrg.* **84**, 1990, 247-255.

[13] C.-L. Chang and L. Povinelli; Piecewise linear approach to the Stokes equations in 3-D. to appear.

[14] P. Ciarlet; *Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[15] L. P. Franca, R. Stenberg; Error analysis of some Galerkin least-squares methods for the elasticity equations, *SIAM J. Numer. Anal.*, **28**, 6, 1991, 1680-1697.

[16] V. Girault and P.-A. Raviart; *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1986.

[17] M. Gunzburger; *Finite Element Methods for Viscous Incompressible Flows*, Academic, Boston, 1989.

[18] M. Gunzburger, M. Mundt, and J. Peterson; Experiences with computational methods for the velocity-vorticity formulation of incompressible viscous flows, *Computational Methods in Viscous Aerodynamics*, Elsevier, Amsterdam, 1990, 231-271.

[19] B.-N. Jiang; A least-squares finite element method for incompressible Navier-Stokes problems, *Inter. J. Numer. Meth. Fluids* **14**, 1992, 843-859.

[20] B.-N. Jiang and C. Chang; Least-squares finite elements for the Stokes problem, *Comput. Meth. Appl. Mech. Engrg.* **78**, 1990, 297-311.

[21] B.-N. Jiang, T. Lin, and L. Povinelli; Large-scale computation of incompressible viscous flow by least-squares finite element method, to appear in *Comput. Meth. Appl. Mech. Engrg.*

[22] B.-N. Jiang and L. Povinelli; Least-squares finite element method for fluid dynamics, *Comput. Meth. Appl. Mech. Engrg.* **81**, 1990, 13-37.

[23] C. Johnson; A mixed finite element method for the Navier-Stokes equations, *RAIRO* M^2AN, **12**, 1978, 335-348.

[24] D. Lefebvre, J. Peraire, and K. Morgan; Least-squares finite element solution of compressible and incompressible flows, *Int. J. Num. Meth. Heat Fluid Flow* **2**, 1992, 99-113.

[25] J.-L. Lions and E. Magenes; *Nonhomogeneous Elliptic Boundary Value Problems and Applications*, Vol. I, Springer, Berlin, 1972.

[26] J. Roitberg and Z. Seftel; A theorem on homeomorphisms for elliptic systems and its applications, *Math. USSR Sbornik*, **7**, 1969, 439-465.

[27] L. Tang and T. Tsang; A least-squares finite element method for time-dependent incompressible flows with thermal convection, to appear in *Int. J. Numer. Meth. Fluids.*

[28] W. Wendland; *Elliptic Systems in the Plane*, Pitman, London, 1979.

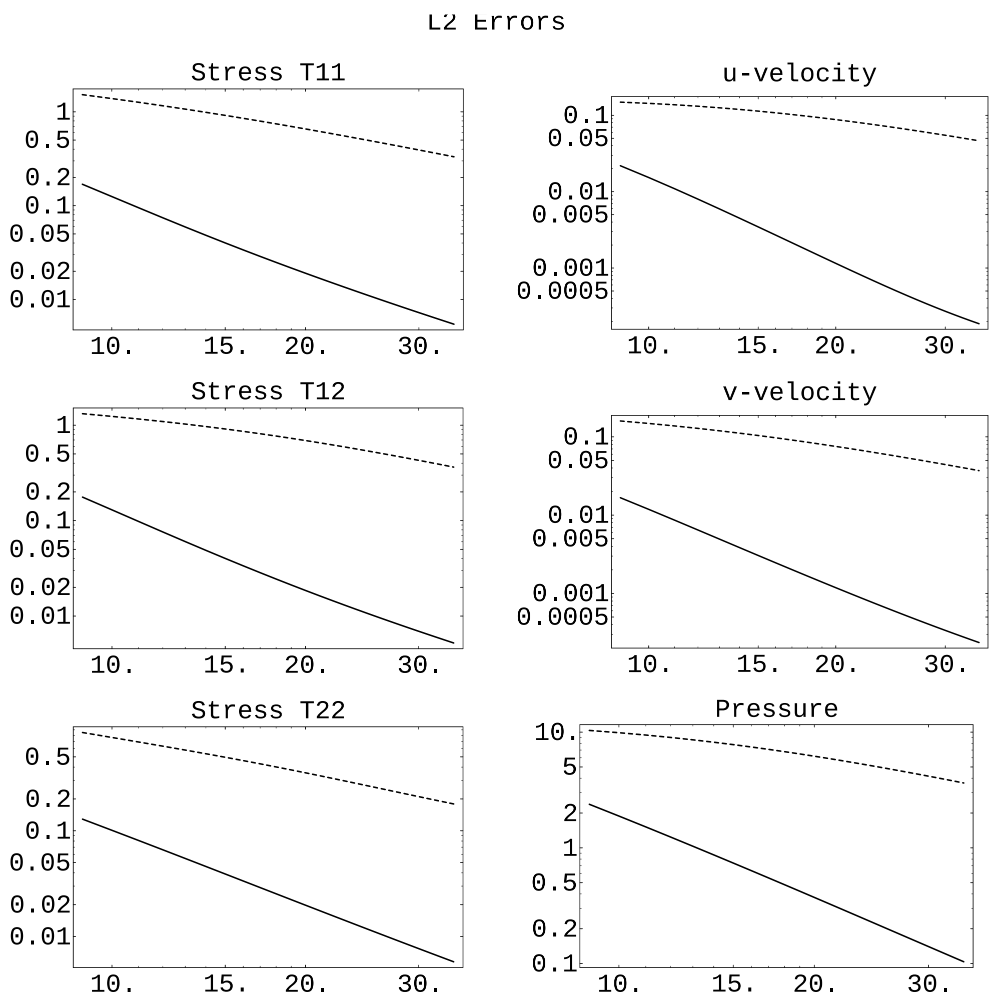

Figure 1: L^2 errors vs. number of grid intervals in each direction. Weighted (solid line) vs. unweighted (dashed line) least-squares method

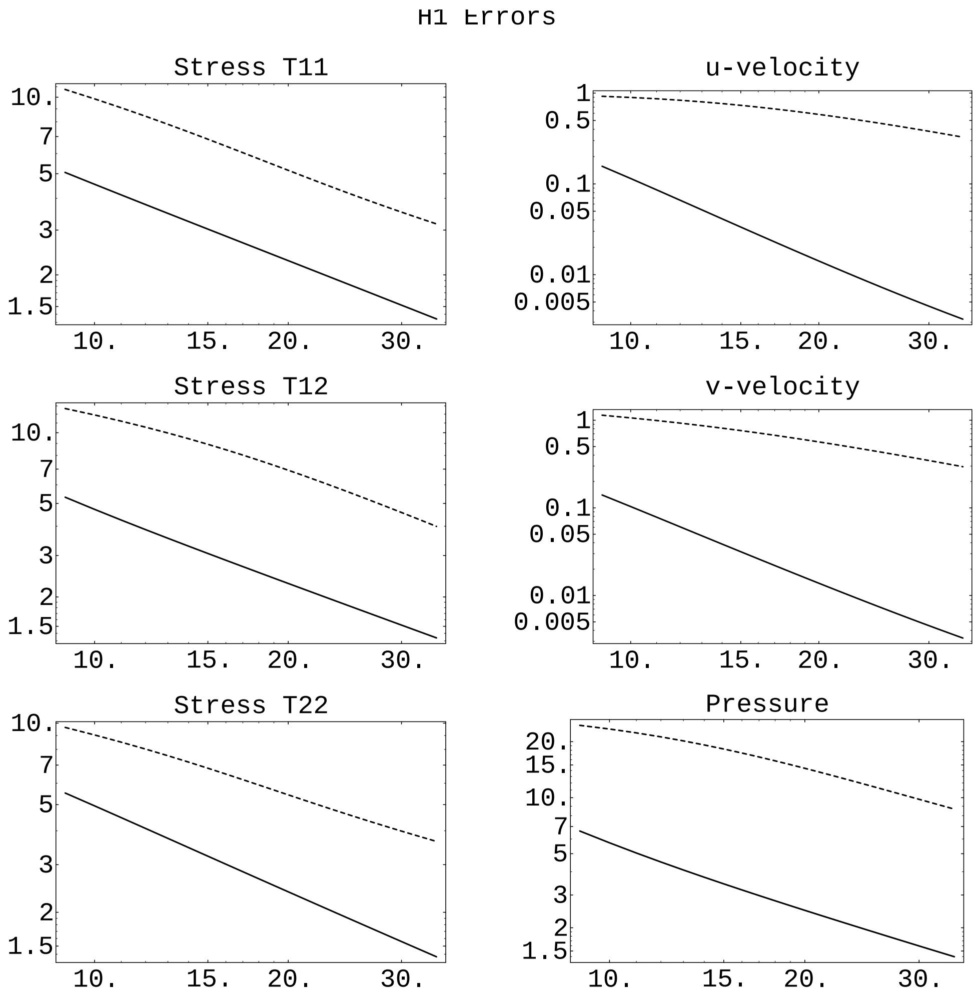

Figure 2: H^1 errors vs. number of grid intervals in each direction. Weighted (solid line) vs. unweighted (dashed line) least-squares method

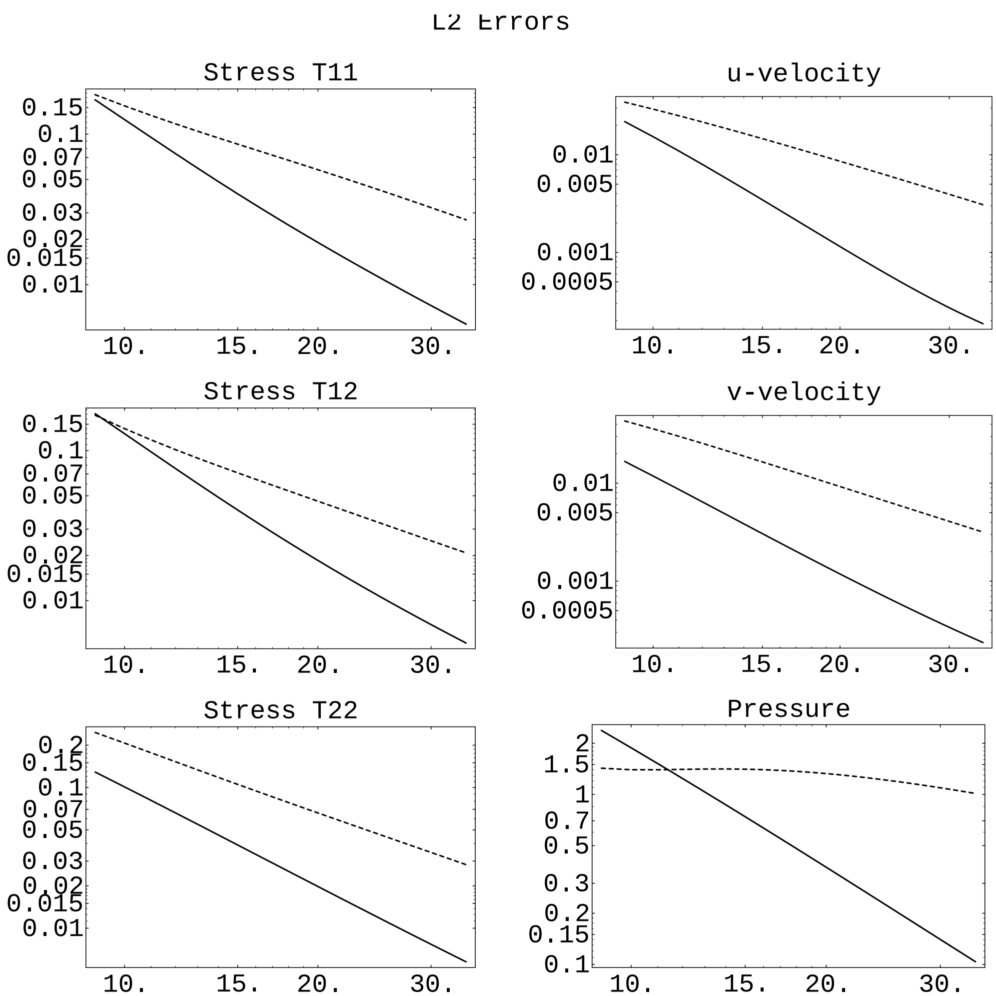

Figure 3: L^2 errors vs. number of grid intervals in each direction. Linear-quadratic (solid line) vs. linear-linear (dashed line) weighted least-squares

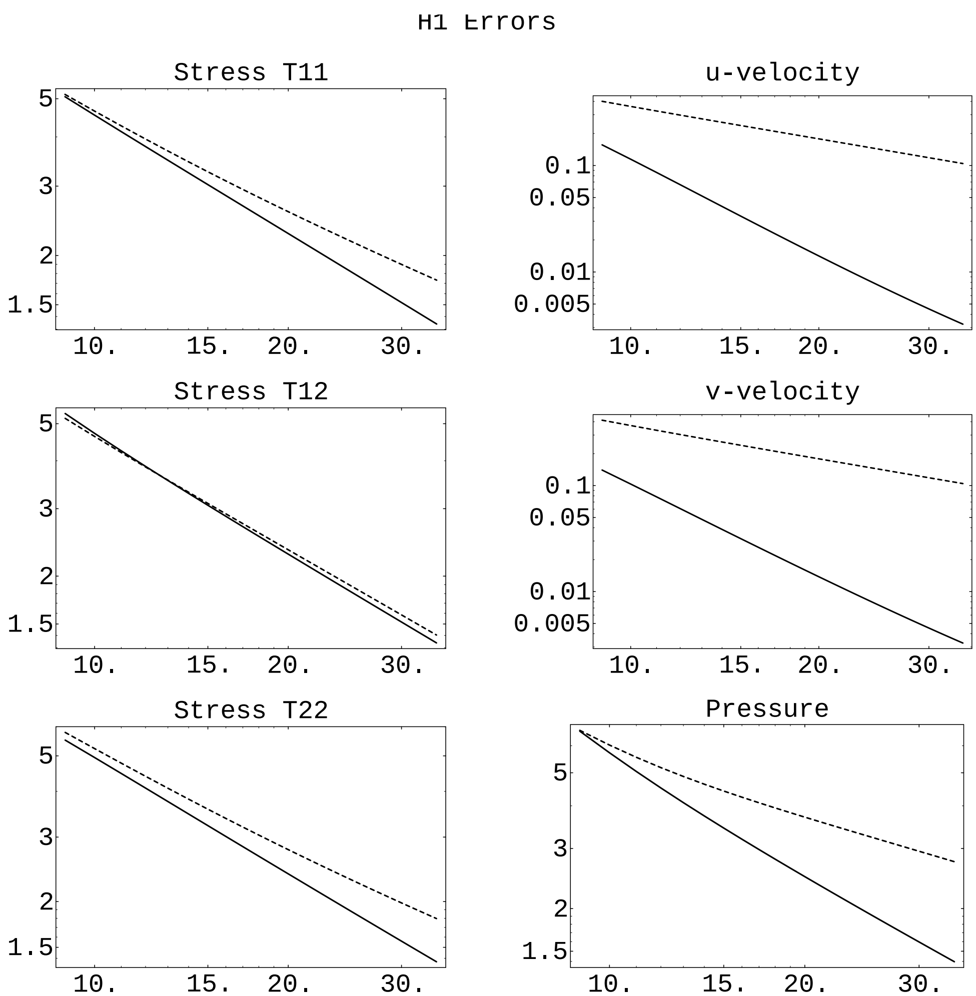

Figure 4: H^1 errors vs. number of grid intervals in each direction. Linear-quadratic (solid line) vs. linear-linear (dashed line) weighted least-squares

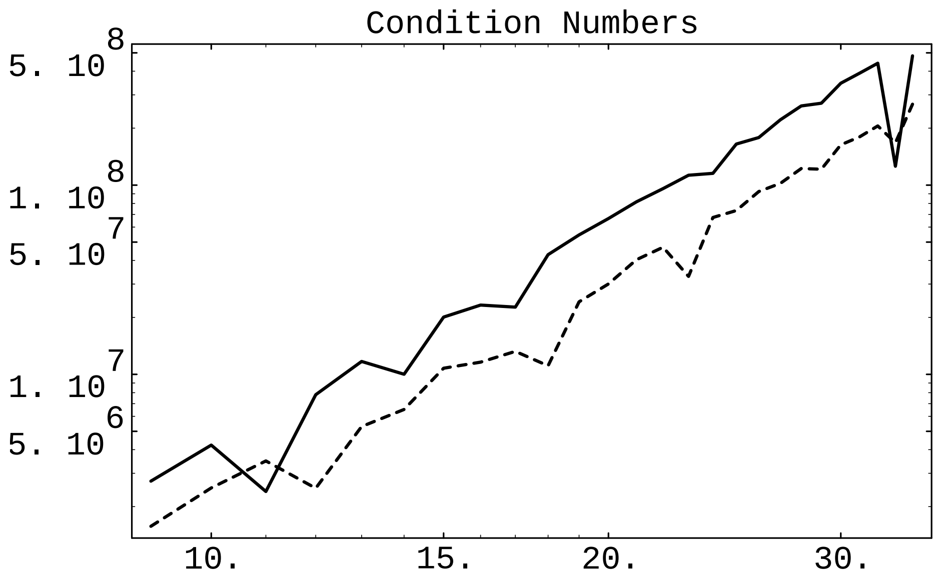

Figure 5: Cond(A): weighted (solid line) vs. unweighted (dashed line) least squares

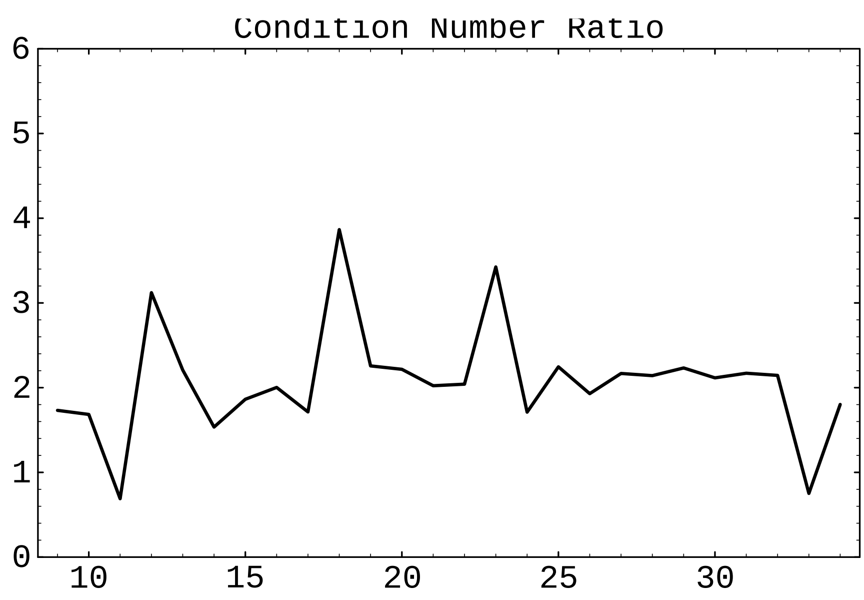

Figure 6: Cond(A) ratio: weighted vs. unweighted least squares

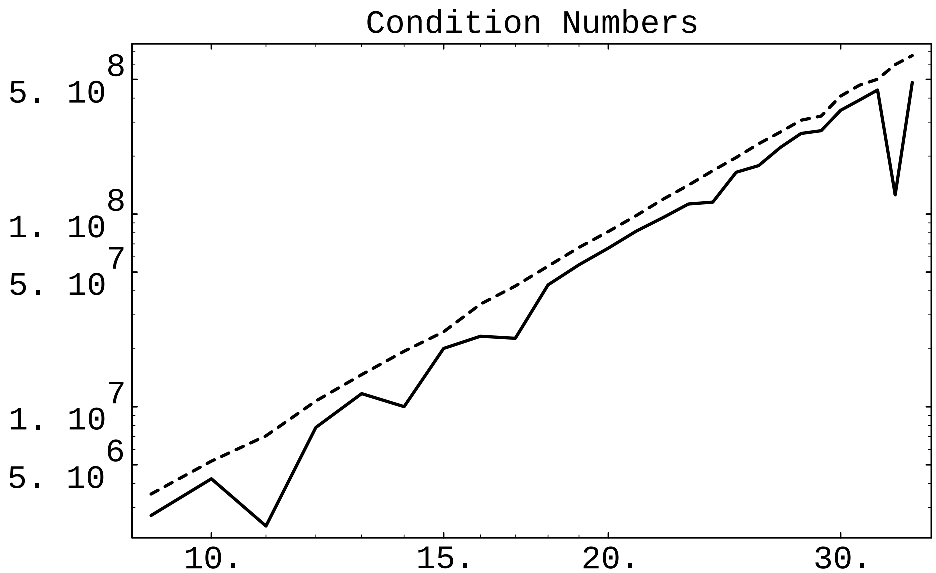

Figure 7: Cond(A): linear-quadratic (solid line) vs. linear-linear (dashed line) least-squares

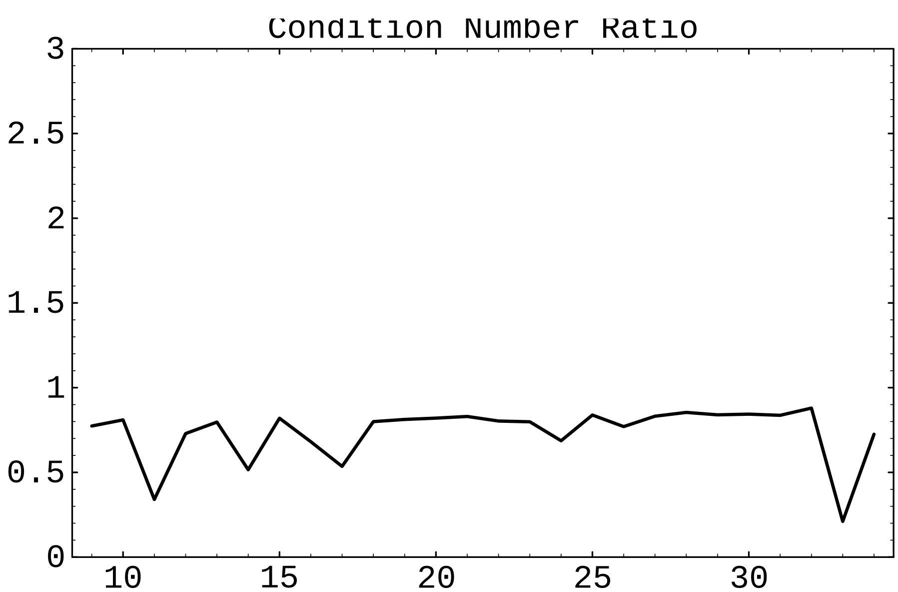

Figure 8: Cond(A) ratio: linear-quadratic vs. linear-linear least-squares